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Against chaos in temperature in mean-field spin-glass models

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Abstract

We study the problem of chaos in temperature in some mean-field spin-glass models by means of a replica computation over a model of coupled systems. We propose a set of solutions of the saddle point equations which are intrinsically non-chaotic and solve a general problem regarding the consistency of their structure. These solutions are relevant in the case of uncoupled systems too. Therefore they imply a non-trivial overlap distribution $P(q_{T1T2})$ between systems at different temperatures. The existence of such solutions is checked to fifth order in an expansion near the critical temperature through highly non-trivial cancellations, while it is proved that a dangerous set of such cancellations holds exactly at all orders in the Sherrington–Kirkpatrick (SK) model. The SK model with soft-spin distribution is also considered, obtaining analogous results. Previous analytical results are discussed.

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1. Introduction

In this paper we shall address by analytical means the problem of the correlations between the equilibrium states at different temperatures of the Sherrington–Kirkpatrick (SK) spin-glass model. At any temperature below the critical one there are infinitely many pure states defined by the local values of the magnetization. The correlation between two states are measured by the overlap $q_{\alpha\beta} = \sum_i m_i^\alpha m_1^\beta$ which, according to the Parisi solution, can take values between zero and some $q_{\rm EA}$, which is the self-overlap of the states [1].

While pure states at a given temperature are correlated, for many years it was believed that states at different temperatures were completely uncorrelated: this is the hypothesis of chaos in temperature. It was originally phrased as a constitutive ingredient of the phenomenological droplet theory [8, 9], because otherwise the growth of domains of correlated phases would give strong cooling rate dependence and would not exhibit the so-called rejuvenation effect.

Contrary to this, real spin glasses are, to a large extent, insensitive to the cooling rate; furthermore, if we let an observable relax at a given temperature in the glassy phase and then lower the temperature we observe that its value skips to a higher value and the system apparently rejuvenates. So everything seems to happen as if there were great changes in the free-energy landscape, i.e. chaos in temperature.

A few years after the first observations of this phenomenon [2], it was discovered that, when the sample is heated back to the original temperature, the observable resumes the value it had immediately before the second quench [3]; therefore, while the first effect suggests that the information belonging to aging at a higher temperature is destroyed upon cooling the system (the chaos effect), the second one implies no information loss at all (the memory effect). The two effects are apparently contradictory and their explanation, particularly from a real-space point of view, attracts great interest [4–6].

At the mean-field level it was shown that the equations of dynamics capture the physics of the two phenomena [7].

From a phase-space point of view it seems rather difficult that a purely chaotic picture of the temperature evolution of the free-energy landscape could account for both chaos and memory effects and this is the main reason for our analysis of the equilibrium states correlations. We must be careful on this point since the theoretical work of the last ten years has raised many questions on the possibility of explaining off-equilibrium dynamics by means of the static mean-field free-energy landscape [10]. At any rate, the ideas arising from the Parisi solution of the mean-field SK model have often proved to be very fruitful, for example in constructing phase-space pictures of dynamics like the traps model [11]. In particular a scenario, which had been suggested as soon as the ultrametric organization of the states was discovered [12], was advocated in order to explain the rejuvenation and memory effects [13]. This scenario deals with the idea that each valley in the free-energy landscape bifurcates into many others when the temperature is lowered, so that rejuvenation is accounted for by equilibration between the newly born valleys, while memory is due to the fact that the topological structure of the states tree is preserved. The explanation of the two effects in this picture is completely different from the chaotic one and is supported by the fact that both rejuvenation and memory have been observed in numerical simulations on an intrinsically non-chaotic model such as the GREM [14].

Our concern here is confined to checking the existence of correlations between equilibrium states at different temperatures. We do not purport to give a full description of the structure, if any, of such correlations but merely to collect evidence that they do exist, in contrast to the chaos hypothesis.

Chaos is known to hold in mean-field models with different magnetic fields, while the case of equal temperatures has been treated by Sompolinsky in an unpublished work cited by Binder and Young in their 1986 review [15]. In that context it is claimed that there is chaos in temperature in mean-field spin glasses; we do not agree with these findings as we shall explain at the end of section 2. Kondor in 1989 addressed a related problem, i.e. the spatial correlations between different temperature states in finite-dimensional spin glasses. Since chaos was generally accepted at the time he was quite surprised in finding such correlations to be infinitely long-ranged at zero-loop order [16], a result that is deeply connected to our own. However, later Kondor and Végsö showed that at one-loop order the correlation length becomes finite [17].

The problem has also been studied by means of numerical simulations [18–20]; our findings are in agreement with the recent work of Marinari and Billoire [21]. The existence of correlations at different temperatures is well shown numerically; the problem is rather that rejuvenation is not seen in simulations of realistic spin-glass models [22]. Correlations between states at different temperatures have recently been studied also within the TAP approach [23].

This paper is organized as follows. In the next section we shall present the model of coupled replicas of a SK spin glass and discuss its relevance to the problem of chaos. In section 3 we shall write down the saddle-point (SP) equations and propose a set of solutions with a certain structure. The validity of such solutions has been checked perturbatively to fifth order in the order parameter and it turned out that they are non-chaotic, i.e. their free energy at this order is the same as in the uncoupled case. We prove that at all orders these solutions, if they exist, are non-chaotic, showing that this property relies on their structure. Therefore the problem is to check whether the SP equations can be solved at all orders with solutions of these kind. To this extent we provide in section 4 an exact result concerning a particular set of cancellations in the SP equations, showing that they are verified at all orders. This result is connected with Kondor zero-loop expansion in replica field theory. However, the possibility of solving the SP equations with the solutions we propose also relies on other concellations, and we are not able to prove that *all* these cancellations actually hold at all orders.

Since the problem is to check whether the solutions we propose actually exist at all orders, in section 5 we investigate a mechanism that could lead to a breakdown of the scheme at higher orders. The essential features of the problem can be explained starting from the SP equations. As we will see in section 3, the SP equations involve three $n \times n$ matrices Q_1, Q_2 and P that will be parametrized in the standard Parisi form. For instance, the SP equation obtained differentiating the free-energy functional with respect to Q_1 has the form

$$\tau_1 Q_{1ab} - B_{ab}(Q_1, Q_2, P) = 0. \tag{1}$$

Now, a necessary condition to solve the previous equation is that the second term B must have the same structure as the first term Q_1: this is what we call *structural consistency*. Translated into the language of Parisi matrices, it means that if the function $q_1(x)$ has a plateau for x greater than some $x_{1\,\max}$ the function $b(x)$ must have a plateau for x greater than the same $x_{1\,\max}$. For the standard Parisi solution it is easily seen that this is true because B is a function only of Q_1, but in our case B depends also on Q_2 and P. If the functions $q_1(x)$, $q_2(x)$ and $p(x)$ have different structures, i.e. they display a plateaux of unequal length located at different positions, as in our case, it is not trivial that $b(Q_1, Q_2, P)(x)$ has the same structure of $q_1(x)$. We will see that, in general, this is not true. However, we will prove that, when the three functions $q_1(x)$, $q_2(x)$ and $p(x)$ have the structure we propose for them, the function $b(x)$ has the same structure of $q_1(x)$, i.e. our solutions fullfil the necessary condition to solve the previous SP equation.

This result is very general and it prevents the solutions we propose from breaking down due to structural inconsistencies. However, it only states that $B_{ab}(Q_1, Q_2, P)$ has the same structure as Q_1, but does not say if it is really possible to tune the three functions $q_1(x)$, $q_2(x)$ and $p(x)$ in such a way that $B_{ab}(Q_1, Q_2, P)$ is exactly equal to $\tau_1 Q_{1ab}$ in order to solve the SP equation (1). The arguments provided in sections 4 and 5 are two necessary conditions for the existence of non-chaotic solutions; they are completely independent since the first is quantitative while the second is qualitative. However, neither of them is sufficient to prove that the solutions actually exist at all orders.

In section 6 the generalization of the SK model with soft spins will be studied, again showing the absence of chaos. In section 7 we give our conclusions. The solutions are reported in the appendix.

2. The model

We consider a system composed of two replicas of a SK spin-glass model constrained to have fixed values of their mutual overlaps. This model was first studied in the case of replicas of

the same temperature [24]; here, we shall use the generalization to replicas with two different temperatures below the critical one [20]. Denoting by S_i^r the ith spin of the rth replica, we fix a constraint

$$q_c = \frac{1}{N}\sum_{i=1}^{N} S_i^1 S_i^2 \tag{2}$$

where N is the total number of spins. Including the temperature difference, the Hamiltonian of the system is

$$H = -\sum_{i<j} J_{ij}(\beta_1 S_i^1 S_j^1 + \beta_2 S_i^2 S_j^2). \tag{3}$$

So we take the same realization of the quenched $\{J_{ij}\}$ for the two systems. They are chosen with Gaussian probability, zero mean and variance $\frac{1}{N}$. The partition function is restricted to those spin configurations that satisfy (2). The constraint (2) is implemented introducing a Lagrange multiplier ϵ:

$$Z = \sum_{\{S_i^1, S_i^2\}} \int_{-i\infty}^{i\infty} \frac{d\epsilon}{2\pi} \exp\left[-H - \epsilon\left(\sum_{i=1}^{N} S_i^1 S_i^2 - N q_c\right)\right]. \tag{4}$$

Instead of fixing the constraint q_c we can consider the partition function corresponding to the following Hamiltonian:

$$H(\epsilon) = -\sum_{i<j} J_{ij}(\beta_1 S_i^1 S_j^1 + \beta_2 S_i^2 S_j^2) - \epsilon \sum_{i=1}^{N} S_i^1 S_i^2. \tag{5}$$

This corresponds to systems coupled by a forcing term which selects configurations with higher overlap. In the thermodynamic limit the two descriptions are obtained one from the other by a Legendre transformation. In particular, defining $F = -\ln Z$ the following relation holds:

$$\epsilon = \frac{\partial F(q_c)}{\partial q_c}. \tag{6}$$

Introducing replicas to average over the disorder we obtain via standard manipulation the average partition function to the power n:

$$\overline{Z^n} = \mathrm{SP}\ \exp\left[\frac{N}{4}\beta_1^2 \operatorname{Tr} Q_1^2 + \frac{N}{4}\beta_2^2 \operatorname{Tr} Q_2^2 + \frac{N}{2}\beta_1\beta_2 \operatorname{Tr} P^2 \right.$$
$$\left. -N \ln Z[\hat{Q}] - N q_c \sum_{\alpha} \epsilon_\alpha - \frac{N}{2}\sum_{\alpha}\left(\frac{\epsilon_\alpha^2}{\beta_1\beta_2} - 2P_{\alpha\alpha}\epsilon_\alpha\right)\right] \tag{7}$$

$$Z[\hat{Q}] = \sum_{\{S_\alpha^1, S_\alpha^2\}} \exp\left[\tfrac{1}{2}\beta_1^2 \sum_{\alpha\beta} Q_{1\alpha\beta} S_\alpha^1 S_\beta^1 + \tfrac{1}{2}\beta_2^2 \sum_{\alpha\beta} Q_{2\alpha\beta} S_\alpha^2 S_\beta^2 + \beta_1\beta_2 \sum_{\alpha\beta} P_{\alpha\beta} S_\alpha^1 S_\beta^2\right] \tag{8}$$

where by SP we mean the value computed at the SP with respect to the set $\{\epsilon_\alpha\}$ and to the order parameter which is a $2n \times 2n$ matrix $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$. The SP equations are then

$$\begin{aligned} Q_{1\alpha\beta} &= \langle S_\alpha^1 S_\beta^1\rangle \qquad Q_{2\alpha\beta} = \langle S_\alpha^2 S_\beta^2\rangle \\ P_{\alpha\beta} &= \langle S_\alpha^1 S_\beta^2\rangle + \frac{1}{\beta_1\beta_2}\epsilon_\alpha\delta_{\alpha\beta} \qquad \epsilon_\alpha = \beta_1\beta_2(P_{\alpha\alpha} - q_c) \end{aligned} \tag{9}$$

where the square brackets mean the average taken with respect to the Hamiltonian

$$H = \tfrac{1}{2}\beta_1^2 \sum_{\alpha\beta} Q_{1\alpha\beta} S_\alpha^1 S_\beta^1 + \tfrac{1}{2}\beta_2^2 \sum_{\alpha\beta} Q_{2\alpha\beta} S_\alpha^2 S_\beta^2 + \beta_1\beta_2 \sum_{\alpha\beta} P_{\alpha\beta} S_\alpha^1 S_\beta^2. \tag{10}$$

The SP equation in the unconstrained case can be obtained by setting ϵ to zero and neglecting the last equation in (9). We eliminate the Lagrange multipliers $\{\epsilon_\alpha\}$ in (7) by replacing their SP values. This gives, for the ϵ-dependent term in $\overline{Z^n}$,

$$-Nq_c \sum_\alpha \epsilon_\alpha - \frac{N}{2} \sum_\alpha \left(\frac{\epsilon_\alpha^2}{\beta_1 \beta_2} - 2P_{\alpha\alpha}\epsilon_\alpha \right) = N \frac{\beta_1 \beta_2}{2} \sum_\alpha (P_{\alpha\alpha} - q_c)^2. \quad (11)$$

To solve the model we need a variational ansatz for the matrices Q_1, Q_2, and P. We choose each of them to be a Parisi hierarchical matrix: in particular this fixes $P_{\alpha\alpha} = p_d$ and $\epsilon_\alpha = \epsilon$ for any replica index α.

Let us comment on the relevance of this model to the problem of chaos. The mathematical formulation of the chaos hypothesis is

$$\overline{\langle S_i \rangle_{T1} \langle S_i \rangle_{T2}} = 0. \quad (12)$$

That is, any two equilibrium states at different temperatures have zero overlap. Now, the absence of chaos would imply that there are pure states at different temperatures with non-zero overlap, so that we would have a non-trivial function $P(q_{T1T2})$ that measures the probability of finding an overlap q between two pure states of systems at different temperatures, weighed according to the Gibbs measure. Turning to our model, we see that, if we choose the constraint inside the support of the function $P(q_{T1T2})$, the only effect will be to select those couples of pure states that satisfy the constraining relation. As the number of pure states grows less than exponentially with N, selection of only some couples of pure states, instead of all, will not change the free energy or other extensive quantities. So, in the absence of chaos we expect to find for some values of the constraint the same values of the free energy as in the unconstrained case (i.e. the sum of the free energies of the two uncoupled systems) while in the presence of chaos the free energy could increase.

So far we have established that an increase in free energy implies chaos, but what if we find no such increase? Let us show that this would imply a nontrivial $P(q_{T1T2})$, i.e. absence of chaos. Suppose we find a set of solutions with the same free energy of the unconstrained case corresponding to values of the constraint q_c spanning an open set from zero to some $p_{\max}$; then from (6) we find that, for all these solutions, the relation $\epsilon = 0$ holds. Looking at the expressions of the free energy and of the SP equations (9) we see that $\epsilon = 0$ implies that these solutions solve the unconstrained two-system problem too. Now, it is well known that the $P(q)$ of the single-system problem can be reconstructed from the matrix Q used to evaluate the free energy [1]. One then finds the following relations:

$$q^{(k)} = \int q^k P(q)\, \mathrm{d}q = \lim_{n\to 0} \overline{Q_{\alpha\beta}^k} \quad (13)$$

where the bar denotes average over all the solutions of the SP equation. Indeed, when replica symmetry is broken there is always more than one solution; given one of them, others can be obtained by permutations of the replica indices. Following the same steps one can show that the function $P(q_{T1T2})$ can be obtained from the order parameter $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$ of the *unconstrained* two-system problem through the relations

$$q_{T1T2}^{(k)} = \int q_{T1T2}^k P(q_{T1T2})\, \mathrm{d}q_{T1T2} = \lim_{n\to 0} \overline{P_{\alpha\beta}^k}. \quad (14)$$

Again the bar means average over all the solutions of the SP equations; we have, of course, the solution $P = 0$, $Q_1 = Q_{1\mathrm{free}}$, $Q_2 = Q_{2\mathrm{free}}$, but, contrary to what is sometimes stated, nothing forbids the existence of solutions with a nonzero P. By looking at (14) we see that we may not know all the solutions or how to average over them but what we do know is that the existence of solutions with $P \neq 0$ implies a nontrivial $P(q_{T1T2})$.

As stated above, solutions of the constrained two-system model with the same free energy of the unconstrained one are solutions of the latter as well, so we can safely say that if these solutions exist there is no chaos in temperature, while, if they do not, pure states at different temperatures are completely uncorrelated.

The fact that non-chaotic solutions of the constrained two-system problem are also solutions of the unconstrained problem is very important. As we saw, by this argument we can safely claim that if these solutions exist the $P(q_{T1T2})$ is non-trivial. Instead it will be dangerous to infer a non-trivial $P(q_{T1T2})$ only from the fact that the free energy per spin in the constrained case is equal to the sum of the free energy of the two uncoupled systems. We shall describe a situation where this recipe could lead to wrong results.

The spin-glass pure states have all the same free energy per spin but their relative weights, due to the correction of order $1/N$, are very different and only a few states are relevant for physical quantities like the $P(q)$.

Now let us imagine a situation of weak chaoticity where the two complete sets of states at different temperatures are strongly correlated but the corrections of order $1/N$ are completely reshuffled: the few relevant states at T_2 are located in a different portion of the phase space from that of the relevant states at T_1. Therefore, in this situation we must have a trivial $P(q_{T1T2})$, i.e. a delta function centred at zero. At the same time we will also have (non-relevant) states with a non-zero overlap between them. A non-zero constraint will select those couples of non-relevant states yielding the same free energy per spin of the unconstrained case, inducing one wrongly to think that the support of the $P(q_{T1T2})$ is non-zero, i.e. that there is no chaos in temperature, while we assumed at the beginning that chaos in temperature, although weak, is present.

The fact that there is a direct connection between the constrained and unconstrained cases ensures that the weak chaoticity picture described above is not possible and that by studying the constrained problem we always obtain sound physical information on the function $P(q_{T1T2})$ of two uncoupled systems.

Let us discuss the results reported in [15]; rephrased in the context we have discussed, the method consists in studying the unconstrained problem, expanding the free-energy functional (7) as follows:

$$F(\hat{Q}) = F^{(0)}(\hat{Q}) + \Delta T F^{(1)}(\hat{Q}) + \Delta T^2 F^{(2)}(\hat{Q}) + \cdots \tag{15}$$

where $F^{(0)}$ is the functional at equal temperatures and ΔT, the temperature difference, is assumed to be small.

The unperturbed equal-temperature case is readily solved: the Parisi solution in its standard single-system form corresponds to $Q_1 = Q_{\text{standard}}$, $Q_2 = Q_{\text{standard}}$, $P = q(0) = 0$. The symmetry between the two systems definitely implies that solutions with a non-zero P can be obtained from the standard one by proper permutations of the replica indices. Now, the perturbing terms in (15) break the symmetry between the two systems and one has to check whether they remove the degeneracy in free energy of the solutions.

The free energy of each solution is evaluated in powers of ΔT substituting in (15) the zero-order equal-temperature solutions; by this procedure a free-energy difference to second order in ΔT is found. However, *to second order one must consider the contribution to F that belongs to the splitting of the zero-order solution*. Explicitly one must add the term

$$\Delta T^2 \frac{\partial F^{(1)}}{\partial \hat{Q}_{ab}} (\mathcal{D}^2 F^{(0)})^{-1}_{ab,cd} \frac{\partial F^{(1)}}{\partial \hat{Q}_{ab}}\bigg|_{\hat{Q}=\hat{Q}^{(0)}} . \tag{16}$$

This term too is not permutationally invariant; therefore, it might cancel the one belonging to the expansion of F. While $F^{(1)}$ has a simple expression evaluating exactly (16) it is impossible with our present knowledge.

However, the whole procedure seems unreliable for more general reasons. For instance, if we try to apply it to the standard single-system case we face the problem of the non-analytical temperature dependence of the Parisi solution $q(x)$ caused by the presence of the plateau. If we think in terms of the hierarchical replica-symmetry breaking (RSB) scheme it can be realized that, by trying to construct a solution starting from a different-temperature one, we obtain a solution with the same breakpoint, i.e. a wrong one.

3. Non-chaotic solutions

Near the critical temperature the order parameter is expected to be small so that one can expand the SP equations (8) in powers of $\hat{Q}$ and then obtain approximate solutions in powers of the reduced temperature $\tau = T_c - T$.

By means of standard manipulations and a proper temperature-dependent rescaling of the order parameter ($\beta_1^2 Q_1 \to Q_1$, $\beta_1^2 Q_2 \to Q_2$, $\beta_1\beta_2 P \to P$) we obtain the free energy up to fifth order

$$\begin{aligned} F(\hat{Q}) = -\lim_{n\to 0}\frac{1}{2}\Bigg\{ & \tau_1 \operatorname{Tr} Q_1^2 + \tau_2 \operatorname{Tr} Q_2^2 + 2\tau_{12}\operatorname{Tr} P^2 + \frac{\omega}{3}\operatorname{Tr}\hat{Q}^3 \\ & + \frac{u}{6}\sum_{ab}\hat{Q}_{ab}^4 + \frac{v}{4}\operatorname{Tr}\hat{Q}^4 - \frac{y}{2}\sum_{abc}\hat{Q}_{ab}^2\hat{Q}_{bc}^2 + \frac{z}{5}\operatorname{Tr}\hat{Q}^5 \\ & - s\sum_{ab}\hat{Q}_{ab}^2(\hat{Q}^3)_{aa} + \frac{2}{3}t\sum_{ab}\hat{Q}_{ab}^3(\hat{Q}^2)_{ab} + \frac{n}{2}(p_d - q_c)^2\Bigg\} \end{aligned} \tag{17}$$

where

$$\tau_1 = \frac{1-T_1^2}{2} \qquad \tau_2 = \frac{1-T_2^2}{2} \qquad \tau_{12} = \frac{1-T_1T_2}{2}. \tag{18}$$

In the SK model we have $\omega = u = v = y = z = s = t = 1$. The only term that explicitly depends on the constraint is $(p_d - q_c)^2 = \epsilon^2$, so that a general strategy often employed is to fix the diagonal term in P and to solve the SP equations belonging to the remaining components in $\hat{Q}$. By this procedure we obtain a solution corresponding to the constraint

$$q_c = p_d - \frac{\partial F_{\text{free}}(\hat{Q})}{\partial p_d}. \tag{19}$$

We skip the expressions of the SP equations expanded to fifth order in Q_1, Q_2 and P. To a lower order they are reported in equations (36) and (37).

A first attempt to solve the equations was to set all the components of P as constant. This ansatz has a positive free-energy cost, i.e. chaos, *independently of the temperature difference*; however, we know that non-chaotic solutions exist for equal temperatures [24]. For this reason we have tried an ansatz for $\hat{Q}$ that could reproduce these solutions in the limit of equal temperatures.

In the case of equal temperatures the solutions of the constrained problem are represented in figure 1; they display full RSB in the matrix P and have a rather simple structure:

$$\begin{aligned} & q_2(x) = q_1(x) \qquad p_d = q_c \qquad \text{for all} \quad x \\ & q_1(x) = p(x) = q_{\text{free}}(2x) \qquad 0 \leqslant x \leqslant \tfrac{1}{2}x_{\text{free}}(p_d) \\ & q_1(x) = p(x) = p_d \qquad \tfrac{1}{2}x_{\text{free}}(p_d) \leqslant x \leqslant x_{\text{free}}(p_d) \\ & q_1(x) = q_{\text{free}}(x) \qquad p(x) = p_d \qquad x_{\text{free}}(p_d) \leqslant x \leqslant 1 \end{aligned} \tag{20}$$

where $q_{\text{free}}(x)$ is the free Parisi solution and $x_{\text{free}}(q)$ is its inverse. The diagonal terms in P are equal to the constraint, i.e. $q_c = p_d$. These solutions exist for any value of q_c in the support

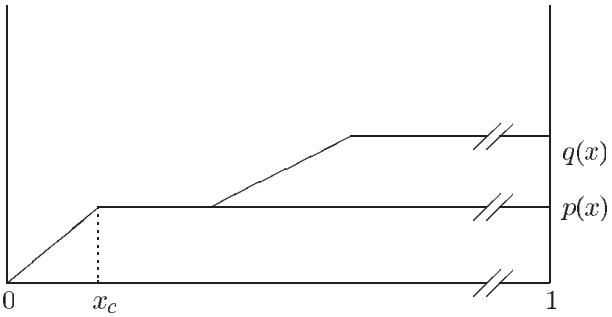

Figure 1. A qualitative sketch of the solutions in the isothermal case: $q_1(x) = q_2(x) = q(x)$ for all x; $p(x) = q(x) = q_{\rm free}(2x)$ for $x < x_{\rm c}$, for $x > x_{\rm c}$ $p(x)$ remains constant and equal to $q_{\rm c}$ while $q(x)$ after an intermediate plateau is joined continuously to $q_{\rm free}(x)$; $q_{\rm c}$ can take values between zero and $q_{\rm EA}$. The diagonal value of P is equal to the constraint, i.e. $p_d = q_{\rm c}$.

of the function $P(q)$ and have exactly the same free energy of the free case; actually, it can be shown that they are particular permutations of the free single-system solution [27].

We want to remark that the structure of these solutions is *intrinsically* non-chaotic due to the equality $p(x) = p_d$ for $x > x_{\rm c}$. Indeed the SP equations are

$$\begin{aligned} &Q_{1\alpha\beta} = \langle S^1_\alpha S^1_\beta \rangle \qquad Q_{2\alpha\beta} = \langle S^2_\alpha S^2_\beta \rangle \\ &P_{\alpha\beta} = \langle S^1_\alpha S^2_\beta \rangle + \frac{1}{\beta_1\beta_2}\epsilon_\alpha\delta_{\alpha\beta} \qquad \epsilon_\alpha = \beta_1\beta_2(P_{\alpha\alpha} - q_{\rm c}). \end{aligned} \tag{21}$$

Now if we have $p(1) = p_d$ we can make the permutation $S^1_1 \leftrightarrow S^1_2$, which is *not* a natural permutation of the Hamiltonian, leaving the matrix $\hat{Q}$ unchanged. Since ϵ appears only in the equation for the diagonal term $P_{\alpha\alpha}$ (i.e. p_d within the Parisi ansatz), this implies

$$p(1) = \langle S^1_1 S^2_2 \rangle = \langle S^1_2 S^2_2 \rangle = p_d \rightarrow \epsilon = 0. \tag{22}$$

So we see that $\frac{\partial F(q_{\rm c})}{\partial q_{\rm c}} = 0$ is true for all these solutions, and noticing that they approach continuously the free solutions in the limit of zero constraint we can safely claim that all these solutions have the same free energy of the free problem.

As stated above, we have looked for solutions approaching continuously (20) in the limit of equal temperatures, so we made some variational attempts with finite RSB for the matrix P. Two facts emerged from this analysis: (i) if we allow $p(1) \neq p_d$ the variational solution is always attracted by a solution with a positive free-energy cost *independently* of the temperature difference, so we discard it; (ii) if we force the variational trial function to have $p(1) = p_d$ we always obtain a negative free-energy cost.

This last result is absurd from a physical point of view since by imposing the constraint we are reducing the configuration space, but in a variational computation it may occur if our trial function does not approach closely enough the true maximum (one of the subtleties of the replica trick is that one has to maximize and not minimize the free energy). The fact that even very complex variational functions showed negative free-energy cost made us suspect that the maximum had zero free-energy cost, so we turned to directly solving the SP equation near the critical temperature.

Having in mind the equal temperature case, we looked at solutions with the structure depicted in figure 2: in the small-x region the three functions $q_1(x), q_2(x)$ and $p(x)$ are all different until they reach the point $x_{\rm c}$ where $p(x) = p_d = q_{\rm c}$; then, for x greater than $x_{\rm c}$, $p(x)$ remains constant while $q_1(x)$ and $q_2(x)$, after an intermediate plateau, are connected continuously to the corresponding free Parisi solutions. These solutions are thought to exist

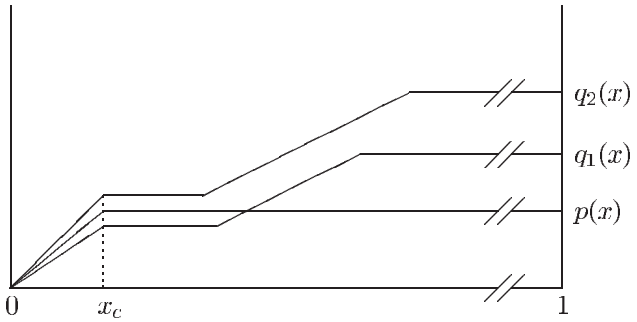

Figure 2. A qualitative sketch of the solutions. In the small-x region they are all different until the point x_c where $p(x) = p_d = q_c$, for $x > x_c$ $p(x)$ is constant and equal to q_c, while $q_1(x)$ and $q_2(x)$, after an intermediate plateau, are joined continuously to the corresponding free function; q_c can take values between zero and some $q_{c\,max}$ at which the two plateaux of the function at the higher temperature merge. At zero order the slope of the functions is 1 in the first region and $1/2$ in the intermediate ones.

for values of the constraint from zero to a maximum value where the two plateaux of the function at the higher temperature merge ($T_1 > T_2$ in figure 2).

We postpone to section 5 the discussion of a consistency problem concerning such a structure for the solutions.

In the isothermal case the functions in the small-x region do not depend on the value of q_c, which only acts as a knife that fixes the position and length of the intermediate plateaux; to the order we compute we cannot say if this is true even for the two-temperature problem, but we believe that it is. It is important to notice that these solutions, provided they actually exist, intrinsically display a zero free-energy cost due to the equality $p(1) = p_d$ as in the isothermal case (see equation (22)).

We checked that, to the fifth order in the expansion in the order parameter, it is possible to find non-chaotic solutions with the structure described above. In the following we sketch the essential features of the calculation.

The expression for the free-energy functional truncated to the fifth order allows us to compute the functions to the second order in the regions before the starting point of the large plateau because these regions already span an interval of order one in the reduced temperatures. Instead, the value of the plateau can be evaluated to the third order as it spans a region of order zero in the reduced temperature.

The SP equations can be solved in terms of the Parisi functions $q_1(x)$, $q_2(x)$ and $p(x)$ expanded in powers of x; one can then use the terms proportional to x^3 and higher to determine the functions in the small-x region and the terms proportional to x to determine the value of the plateau. We determined the functions in the small-x region and then checked such functions joined with the free Parisi solutions to satisfy the equations.

To the fifth order in $\hat{Q}$ the SP equations display terms proportional to x and x^3, plus constant terms due to the presence of the intermediate plateau for $x > x_c$. Having determined the functions in the small-x region using the x^3 terms we had to check nine coefficients to be zero: three proportional to x, one for each of the three equations for $q_1(x)$, $q_2(x)$ and $p(x)$ in the small-x region; two proportional to x in the regions of the intermediate x from the two equations for $q_1(x)$, $q_2(x)$; and four belonging to the same regions corresponding to the x^3 and constant terms.

After a tedious but straightforward computation all these coefficients turned out to be zero. However, while the coefficients of $x, x^3, \ldots$ in the equations for $q_1(x)$ and $q_2(x)$ are

null, independently of the value of $\omega, u, v, y \ldots$, the corresponding coefficients in the equation for $p(x)$ only cancel provided the following relationships hold:

$$1 - \frac{v}{\omega^2} = 0 \qquad 1 - \frac{3v^2}{\omega^4} + \frac{2z}{\omega^3} - \frac{2vy}{\omega^4} + \frac{2s}{\omega^3} = 0. \tag{23}$$

The constant terms in these relationships are the coefficients of $(\tau_1 - \tau_2)^2$ and $(\tau_1 + \tau_2)(\tau_1 - \tau_2)^2$ in the expansion of $4\tau_{12}$.

The free-energy difference has been evaluated either directly on the maximum and both thermodynamically integrating the energies (that have a much simpler expression in the order parameter) with respect to the temperatures. It turned out to be proportional to the left-hand sides of (23), so that it is zero when relations (23) hold. This is consistent with the previous statement that the structure of the solutions is intrinsically non-chaotic; indeed, it implies that we cannot choose the coefficients $\omega, u, v, y \ldots$ to have a positive free-energy cost and at the same time to satisfy the SP equations.

The expressions of $q_1(x), q_2(x)$ and $p(x)$ in terms of $\omega, u, v, y \ldots$ are reported in the appendix.

4. Exact results: the constraint-independent cancellations

In this section we prove that a particular set of cancellations in the SP equation of $p(x)$ is verified at all orders in the SK model. Expanding the equation in powers of x, we must find that the coefficient of each power equals zero. We concentrate on the coefficient of x, and in particular on those terms in the coefficient of x that do not depend of the value of the constraint q_c, but only on the temperatures of the two systems. These terms must sum up to zero. It will be proven that these constraint-independent cancellations hold at any order in the SK model. We will see that these cancellations are the origin of the relationship (23); this means that going to higher order we will encounter relations between the coefficient like (23) such that they are verified by the corresponding coefficients of the SK model.

One should not forget that to verify the SP equations, other cancellations should be checked at all orders, while they have been checked only to lowest orders in the computation reported above; however, it is interesting to notice that at the order we computed we found that these cancellations hold independently of the parameters $\omega, u, v, y \ldots$ so one may conjecture that at higher orders they can always be fullfilled by properly tuning the functions in the small-x region. According to this conjecture, for any given spin-glass model the only relevant condition for the existence of the non-chaotic solutions is that the constraint-independent cancellations we consider here hold at all orders.

The free energy (17) can be written as the sum of the free energies of the free problem plus a term proportional to P:

$$\begin{aligned} F(P) = -\lim_{n\to 0} \frac{1}{2n} \Bigg\{ & \operatorname{Tr}(AP^2) + \frac{u}{3} \sum_{ab} P_{ab}^4 + \frac{v}{2} \operatorname{Tr} P^4 - y \sum_{abc} P_{ab}^2 P_{bc}^2 \\ & + z \operatorname{Tr} P^4 (Q_1 + Q_2) - \frac{3s}{n} \operatorname{Tr} P^2 (Q_1 + Q_2) \operatorname{Tr} P^2 \\ & + \tfrac{4}{3} t \sum_{ab} P_{ab}^3 (PQ_1 + PQ_2)_{ab} \Bigg\} \end{aligned} \tag{24}$$

$$\begin{aligned} A_{ab} = & \left(2\tau_{12} - \frac{y}{n} (\operatorname{Tr} Q_1^2 + \operatorname{Tr} Q_2^2) - \frac{s}{n} (\operatorname{Tr} Q_1^3 + \operatorname{Tr} Q_2^3) \right) \delta_{ab} + \omega (Q_1 + Q_2)_{ab} \\ & + v (Q_1^2 + Q_2^2 + Q_1 Q_2)_{ab} + z (Q_1^3 + Q_2^3 + Q_1 Q_2^2 + Q_2 Q_1^2)_{ab} \end{aligned}$$

$$-\frac{s}{n}(\mathrm{Tr}\, Q_1^2)(2Q_1+Q_2)_{ab}-\frac{s}{n}(\mathrm{Tr}\, Q_2^2)(2Q_2+Q_1)_{ab}+\frac{2}{3}t(Q_{1ab}^3+Q_{2ab}^3). \tag{25}$$

The equation one obtains differentiating with respect to P is

$$-\frac{\partial F(P)}{\partial P_{ab}}=(AP)_{ab}+\tfrac{2}{3}uP_{ab}^3+v(P^3)_{ab}-yP_{ab}((P^2)_{aa}+(P^2)_{bb})+\tfrac{1}{2}\epsilon\,\delta_{ab}=0. \tag{26}$$

In the last equation we neglected the fourth order terms which contain powers of P greater than one, as in reality their order is higher. We are assuming that q_c is of the order of magnitude of the self-overlap in the free case, i.e. of the order of τ_1 and τ_2. The structure of the matrix P (explicitly the equality $p_d = p(1)$) implies that, even if P is of order one, its powers P_n are of order $2n-1$ and not of order n. Being $(p_d^{(n)}, p^{(n)}(x))$ the Parisi function associated with P^n, we know that the quantity $p_d^{(n)}-\bar{p}^{(n)}$ equals $(p_d-\bar{p})^n$, i.e. it is proportional to q_c^{2n}. Now $p^{(n)}(x)$ is different from $p_d^{(n)}$ only on a region of order one and we see immediately that P_n must be of order $2n-1$ in q_c.

Expressing (26) in $q_1(x), q_2(x), p(x)$ and expanding in powers of x we have terms proportional to x and x^3 in the small-x region. We determined the function $p(x)$ using the x^3 terms and thus we must check the x ones to be zero. These are generated by $(AP)_{ab}$ and $P_{ab}((P^2)_{aa}+(P^2)_{bb})$. Then the equation we have to check is

$$(a_d-\bar{a})p(x)+(p_d-\bar{p})a(x)+2(p_d^2-\bar{p^2})p(x)=0 \tag{27}$$

where $(a_d, a(x))$ is the Parisi function associated with the matrix A defined in (25). This equation must really hold only for the terms proportional to x and we may more correctly write it substituting $q(x)$ and $p(x)$ with their derivative in zero, eliminating the x dependence.

The lhs of (27) displays terms explicitly proportional to powers of q_c and others that depend *only* on τ_1 and τ_2; the former terms cancel irrespective of the values of $\omega, u, v, y\ldots$ while the latter sum up to zero, provided that the relations (23) hold.

Since at zero order we have $p'(0)=1$ it is easily seen that these constraint-independent terms belong only to $(a_d-\overline{a})$. It is quite natural that their cancellation depends crucially on the structure of the model (i.e. on the coefficients $\omega, u, v, y\ldots$) because they cannot be controlled by properly tuning the solutions in the small-x region. For completeness we report the q_c-independent terms belonging to the quantity $a_d-\bar{a}$ evaluated directly from (25) which is the origin of the relationships (23)

$$a_d-\bar{a}=\frac{1}{2}\left(1-\frac{v}{\omega^2}\right)(\tau_1-\tau_2)^2+\frac{1}{2}\left(1-\frac{3v^2}{\omega^4}+\frac{2z}{\omega^3}-\frac{2vy}{\omega^4}+\frac{2s}{\omega^3}\right)(\tau_1+\tau_2)(\tau_1-\tau_2)^2. \tag{28}$$

We will prove that the constraint-independent terms in $a_d-\bar{a}$ are null at all orders in the SK model.

According to the previous discussion the matrix A is defined as the sum of all the P-independent matrices X that appears in the free-energy functional in the form $\mathrm{Tr}\, XP^2$ (actually the expression (25) is truncated to third order in the order parameter, because it is derived from (17) which is in turn the free-energy functional truncated to the fifth order). Therefore A is related to the second derivative of F with respect to P. By direct inspection one verifies that $A_{am}\delta_{bn}$ is equal to $\partial^2 F/\partial P_{ab}\partial P_{mn}$ plus terms that depend explicitly on P. We are interested in the q_c-independent part of A so we can consider the limit $q_c\to 0$ where they become equal. Therefore we have to compute $\partial^2 F/\partial P_{ab}\partial P_{mn}$ in the limit of zero constraint (i.e. $P\to 0$). The solutions reduce continuously to the free ones in this limit so we can safely compute it for $P=0$.

Since $\partial^2 F/\partial P_{ab}\partial P_{mn}$ is a component of the Hessian of F, it is the main ingredient to calculate the spatial correlation functions between states at different temperatures of a finite-dimensional spin glass in a Gaussian approximation around mean-field theory. In this context it has been previously computed by Kondor [16]; therefore *the particular cancellations we are dealing with are exactly verified at all orders for much the same reason that at zero-loop order the overlap of the spin correlations at two different temperatures is infinitely long ranged, whatever the difference between them.*

One can easily convince himself that, when $P = 0$, $(a_d - \bar{a})$ is proportional to the minimum eigenvalue of $\partial^2 F/\partial P_{ab}\partial P_{mn}$, i.e. to the inverse of the correlation length. Therefore the computation we shall sketch below is a completely equivalent rephrasing of Kondor's computation.

Let us proceed to evaluate $(a_d - \bar{a})$; following Kondor, we start noticing that the Hessian is related to the four-point connected correlation functions

$$nT_1T_2A_{am}\delta_{bn} = nT_1T_2\frac{\partial^2 F}{\partial P_{ab}\partial P_{mn}} = \delta_{ab}\delta_{mn} - \frac{1}{T_1T_2}\left(\langle S_1^a S_2^b S_1^m S_2^n\rangle - \langle S_1^a S_2^b\rangle\langle S_1^m S_2^n\rangle\right). \tag{29}$$

The first identity holds because P is zero; exploiting again this relation we can evaluate the rhs of (29):

$$\begin{aligned} nT_1T_2A_{am}\delta_{bn} = {} & \delta_{ab}\delta_{mn} - \frac{1}{T_1T_2}(\delta_{am}\delta_{bn} + delta_{am}(1-\delta_{bn})Q_{2bn} \\ & + \delta_{bn}(1-\delta_{am})Q_{1am} + (1-\delta_{am})(1-\delta_{bn})Q_{2bn}Q_{1am}) \end{aligned} \tag{30}$$

$$A_{ab} = \sum_{mn}(A_{am}\delta_{bn})\delta_{mn} = \frac{1}{T_1^2T_2^2}(T_1T_2 - 1 - Q_{1ab} - Q_{2ab} - (Q_1Q_2)_{ab}). \tag{31}$$

Now, given a generic ultrametric matrix A, the quantity $a_d - \bar{a}$ is the eigenvalue corresponding to the eigenvector with constant coordinates so we have

$$a_d - \bar{a} = \frac{1}{T_1^2T_2^2}(T_1T_2 - 1 + \bar{q}_1 + \bar{q}_2 - (\bar{q}_1)(\bar{q}_2)) = 0 \tag{32}$$

where in the last identity we used the exact relation [26][1] $\bar{q}_{\text{Parisi}} = 1 - T$. Note that this relation holds because we have assumed that in the large-x region the functions $q_1(x)$ and $q_2(x)$ are exactly equal to the standard Parisi solutions at the corresponding temperatures.

5. Structural consistency

In this section we discuss a problem of consistency related to the structure we propose for the solutions. The essential features of the problem have been presented in the introduction. It can be view as a necessary, but not sufficient, condition to solve the SP equations. We will show that this necessary condition is fullfilled at all orders by the solutions we propose. As said in the introduction this result is completely independent of the result of the previous section.

In the following we will use the SP equations expanded to the lowest order sufficient to clarify the nature of the problem but the results we will obtain are valid at all orders. The SP equations for the single system problem are

$$2\tau_1 Q_{ab} + (Q^2)_{ab} + \tfrac{2}{3}Q^3_{ab} + (Q^3)_{ab} - Q_{ab}((Q^2)_{aa} + (Q^2)_{bb}) + \cdots = 0. \tag{33}$$

The lhs is a sum of all the possible covariants of Q. A covariant is a two-index object built in a permutational covariant way from the matrix Q; given any permutation π between the replica indices the mathematical definition of a covariant M is

$$M(\pi Q) = \pi M(Q). \tag{34}$$

[1] This relation is equivalent to Sompolinsky's $\chi(0) = 1$ in [26].

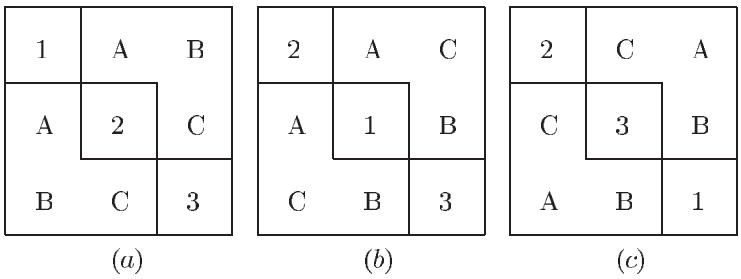

Figure 3. (*a*) A simple hierarchical matrix composed of nine blocks of the same size with $1 = 2 = 3 = q_{\mathrm{D}}$ and $A = B = C = q_{\mathrm{OD}}$. The matrix is invariant under block permutations like those represented in (*b*), (*c*) they can be used to prove separability.

If the matrix Q is a hierarchical matrix, all its covariants are hierarchical matrices with the same structure of Q, i.e. with the same set of block indexes m's; for instance, if the Parisi function $q(x)$ associated with Q has a plateau for x greater than some x_1 the Parisi functions associated with any of its covariants will display a plateau in the same region.

This property of the Parisi matrices can be understood by looking at figure 3: we have a simple hierarchical matrix Q composed of nine blocks of the same size; the components of a single block are all equals, the three blocks on the diagonal are equal to q_{D} and the six blocks off the diagonal are equal to q_{OD}.

In figure 3 two block permutations are also represented which leave the matrix Q unchanged; these permutations can be used to prove that any covariant of Q has its same structure. For instance, if we consider the first permutation and apply to it the definition of covariant (34) together with the fact that this permutation verifies $\pi(Q) = Q$, we obtain that $\pi M = M$. Since this permutation exchanges blocks 1 and 2, the blocks corresponding to 1 and 2 in M are equal, as the blocks corresponding to B and C; using other permutations of this kind we can prove that any covariant M is a hierarchical matrix with the same structure of Q.

This property goes under the name of separability or overlap equivalence [25] because of its physical meaning; it is not clear if it is a condition to solve the SP equation, but, together with stochastic stability, it is likely to be the origin of ultrametricity.

The SP equation for two systems at different temperatures is

$$2\begin{pmatrix} \tau_1 Q_1 & \tau_{12} P \\ \tau_{12} P & \tau_2 Q_2 \end{pmatrix}_{ab} + (\hat{Q}^2)_{ab} + \tfrac{2}{3}\hat{Q}^3_{ab} + (\hat{Q}^3)_{ab} + \cdots = 0 \tag{35}$$

where the indices (a, b) range from 1 to $2n$. All the terms—excluding the first—in the previous equation are identical to those of the SP equation for a single system (33), provided one substitutes the covariants of Q with the corresponding covariants of $\hat{Q}$. Expressing the covariants of $\hat{Q}$ in terms of Q_1, Q_2 and P the previous equation is

$$\begin{aligned} 2\begin{pmatrix} \tau_1 Q_{1\,ab} & \tau_{12} P_{ab} \\ \tau_{12} P_{ab} & \tau_2 Q_{2\,ab} \end{pmatrix} &+ \begin{pmatrix} (Q_1^2 + P^2)_{ab} & (P(Q_1 + Q_2))_{ab} \\ (P(Q_1 + Q_2))_{ab} & (Q_2^2)_{ab} \end{pmatrix} + \tfrac{2}{3}\begin{pmatrix} Q^3_{1\,ab} & P^3_{ab} \\ P^3_{ab} & Q^3_{2\,ab} \end{pmatrix} \\ &+ \begin{pmatrix} (Q_1^3 + P^2(2Q_1 + Q_2))_{ab} & (P^3 + P(Q_1^2 + Q_2^2 + Q_1 Q_2))_{ab} \\ (P^3 + P(Q_1^2 + Q_2^2 + Q_1 Q_2))_{ab} & (Q_2^3 + P^2(2Q_2 + Q_1))_{ab} \end{pmatrix} \\ &+ \cdots = 0 \end{aligned} \tag{36}$$

where the indices (a, b) range from 1 to n. According to the previous equations we have that the equation for Q_1 is

$$2\tau_1 Q_{1ab} + (Q_1^2)_{ab} + P^2_{ab} + \tfrac{2}{3}Q^3_{1ab} + (Q_1^3)_{ab} + ((2Q_1 + Q_2)P^2)_{ab} + \cdots = 0. \tag{37}$$

Notice that P always appears in even power in the invariants of the free energy. This is connected with the fact that each mute index of $\hat{Q}$ must appear an even number of times in zero magnetic field.

The consistency problem we want to address is that the previous equations do not admit a solution for any parametrization of the matrices Q_1, Q_2 and P.

Let us start considering the simplest case where Q_1, Q_2 and P are 1RSB Parisi matrices with different breaking points. If we want to solve (37) to second order we must consider its first three terms. The first two are 1RSB matrices with breaking point x_{Q_1}, while the second is a 1RSB matrix with breaking point x_P. If they are to sum up to zero the only two possibilities are: (a) $P = 0$, which is the trivial one, or (b) $x_{Q_1} = x_P$.

Going to the next order we encounter the last term in which Q_2 also appears, and again, if we want a solution with $P \neq 0$, we must impose $x_{Q_2} = x_{Q_1} = x_P$. Therefore, before solving (37) we already know that the only consistent non-trivial solution must have all equal breaking points.

Generalizing to an arbitrary number k of RSB steps we found that a consistent parametrization for the matrices Q_1, Q_2 and P can be obtained fixing

$$m_i^{Q_1} = m_i^{Q_2} = m_i^P \qquad \forall\, i = 1, \ldots, k. \tag{38}$$

In the limit of infinite RSB steps the three matrices are parametrized by the functions $q_1(x), q_2(x)$ and $p(x)$ which are continuous and can eventually display constant parts (plateaux). In this case the consistent parametrization (38) corresponds to imposing that, if one of the three function displays a plateau, the other two functions must display a plateau of the same length located at the same position.

Looking at figure 2 it is clear that our parametrization is not of the type described; thus, we must check its structural consistency. To clarify what kind of problems we may encounter with such a structure for the solutions let us go back to equation (37).

To second order we must retain the first three terms. For $x < x_c$ we have no problems because all the three terms are varying, for $x > x_c$ P^2 become constant but, as Q_1^2 has the same structure of Q_1, we should reasonably be able to solve the equations.

The problems arise at third order due to the presence of the last term which is proportional to Q_2. Let us consider the region of the second plateau of the function $q_1(x)$ in figure 2. In this region all the terms in (37) that depend on P and Q_1 are constant because $q_1(x)$ and $p(x)$ are constant. Instead the last term $(2Q_1 + Q_2)P^2$ that depends on Q_2 will vary in general until the point $x_{2\max}$ where the second plateau of $q_2(x)$ starts.

Clearly if $(2Q_1 + Q_2)P^2$ varies, equation (37) cannot be verified in both the two regions $x_{1\max} < x < x_{2\max}$ and $x_{2\max} < x < 1$.

The same problem is present in the region between the ending point of the intermediate plateaux of $q_1(x)$ and $q_2(x)$, in which as one function varies the other is constant.

In a few words, the problem is that the equation for $q_1(x)$ displays terms depending on $q_2(x)$ which, for $x > x_c$, have a completely different structure. The same problem is present for the equation of $q_2(x)$ and $p(x)$.

In the following it is shown that the equality $p(1) = p_d$ allows us to solve these problems of structural consistency at all orders. For instance, it implies that all the terms of the form $P^{2p} Q_1^q Q_2^r$ display a plateau for $x > x_c$ so that any dependence on the structure of $q_1(x)$ and $q_2(x)$ in this region is removed.

The last statement can be checked by direct inspection evaluating the product AP, given that P satisfies $p(x) = p_d$ for $x > x_c$ and A is a generic Parisi matrix. It turns out that this product has the same structure of P, i.e. we have $(ap)(x) = (ap)_d$ for $x > x_c$.

In general, through $p(x) = p_d$ the lhs of (37) can be recast order by order as a sum of

Parisi matrices that have exactly the same structure of Q_1, even if they depend on P and Q_2. The same is true for the SP equation of Q_2 and P.

These recast terms are just the corresponding components of the covariant of $\hat{Q}$. For instance, in the equation for Q_1 the corresponding component of the covariant $\hat{Q}^3$ is $(Q_1^3 + P^2(2Q_1 + Q_2))$. The first term has the structure of Q_1 with two plateaux located at the same position while the second is constant for $x > x_c$; consequently $(Q_1^3 + P^2(2Q_1 + Q_2))$ has the structure of Q_1.

In other words, structural consistency is ensured because when $p(1) = p_d$ the matrix $\hat{Q}$ is separable. Writing one of its covariants $M(\hat{Q})$ as

$$M(\hat{Q}) = \begin{pmatrix} M_1(Q_1, Q_2, P) & M_{12}(Q_1, Q_2, P) \\ M_{12}(Q_1, Q_2, P) & M_2(Q_1, Q_2, P) \end{pmatrix} \tag{39}$$

separability for $\hat{Q}$ means that $M_1(Q_1, Q_2, P)$ has the same block structure of Q_1, $M_2(Q_1, Q_2, P)$ has the same block structure of Q_2 and $M_{12}(Q_1, Q_2, P)$ has the same block structure of P.

In general, a matrix $\hat{Q}$ composed of three hierarchical matrix Q_1, Q_2 and P is not globally separable; a sufficient condition for this is that the three matrices have the same structure, i.e. that they verify equations (38). This however is not our case.

To prove that the equality $p(1) = p_d$ is a sufficient condition for the separability of $\hat{Q}$ let us take a look at figure 4. Here the matrix $\hat{Q} = \begin{pmatrix} Q_1 & P \\ P^t & Q_2 \end{pmatrix}$ is sketched qualitatively according to figure 2.

The block index m_c in figure 4 corresponds to x_c in figure 2. For $m_c < m < n$ $(0 < x < x_c)$ the three matrices Q_1, Q_2 and P have the same structure, i.e. the same set of m's (for simplicity in (4) there is only the intermediate index m_i between m_c and n).

For $1 < m < m_c$ $(x_c < x < 1)$ the n/m_c blocks of size m_c in Q_1 have an internal structure different from the corresponding n/m_c blocks in Q_2, while the n/m_c blocks in P have all their components equal to p_d.

In particular the crossed blocks in Q_1 are identical to the blocks of the same size in the standard Parisi solution at temperature T_1; those in Q_2 correspond to the standard solution at T_2.

If we consider one of the blocks of size m_c in the matrix Q_1, say the first, we can make permutations like those of figure 3 between its inner blocks leaving Q_1 unchanged. The invariance under this block permutation ensures that any covariant of Q_1 has the same structure of Q_1.

Now, to a block of size m_c in Q_1 corresponds a block of size m_c on the diagonal of P *components of which are all equal to* p_d, this ensures that not only Q_1 but also the global matrix $\hat{Q}$ is left unchanged by the same block permutations.

We recall how separability follows from the invariance under these block permutations. Invariance means $\pi\hat{Q} = \hat{Q}$, so the definition of covariant, i.e. $M(\pi\hat{Q}) = \pi M(\hat{Q})$, implies $\pi M(\hat{Q}) = M(\hat{Q})$. These permutations exchange blocks of $M(\hat{Q})$, so that the last equality means that these blocks are equal and $M(\hat{Q})$ have the same structure of $\hat{Q}$.

In other words, the structure of a matrix like $\hat{Q}$ or $M(\hat{Q})$ is unequivocally determined by the set of block permutations that leave it unchanged, and the equality $p(1) = p_d$ implies that the set of invariant block permutations of $M(\hat{Q})$ coincide with that of $\hat{Q}$.

Instead, if $p_d \neq p(1)$, in order to leave P unchanged under the permutations between the internal blocks of, say, the first block of size m_c of Q_1, we should make the same block permutations on Q_2. Since Q_2 has a different block structure from Q_1, it would not be left unchanged.

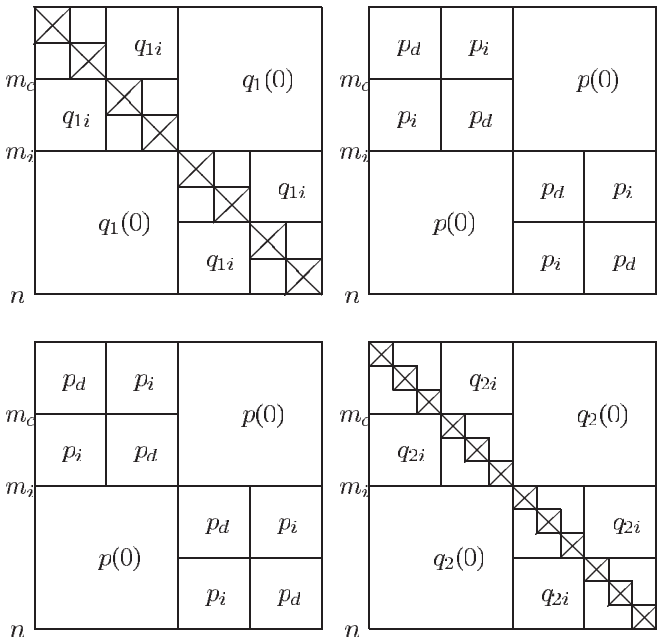

Figure 4. The global matrix $\hat{Q}$. The block index m_c corresponds to x_c in figure 2. For $m_c < m < n$ ($0 < x < x_c$) the three matrices Q_1, Q_2 and P have the same structure, i.e. m_i is identical for the three matrices. For $1 < m < m_c$ ($x_c < x < 1$) the n/m_c blocks of size m_c in Q_1 have a different internal structure from the corresponding n/m_c blocks in Q_2, while the n/m_c blocks in P have all their components equal to p_d.

6. The SK model with soft-spin distribution

In this section we shall apply the approach of coupled replicas to the generalization of the SK model to soft spin distribution. This model can both be mapped onto the SK model by a proper redefinition of the parameters $\omega, u, v, y \ldots$ appearing in the free-energy functional (17).

In the case of continuous spins each invariant belonging to the term ln Tr exp $\left[\sum_{ab} \hat{Q} S_a S_b\right]$ in the free-energy functional must be multiplied by a proper product of cumulants of the soft-spin distribution. Rescaling the order parameter by a factor $\langle S^2 \rangle$ the mapping goes as follows:

$$\begin{aligned} &\omega \to \omega = 1 \qquad v \to v = 1 \qquad u \to u = \frac{\langle S^4 \rangle^2}{\langle S^2 \rangle^4} \\ &y \to y = \frac{\langle S^4 \rangle}{\langle S^2 \rangle^2} \qquad z \to z = 1 \qquad s \to s = \frac{\langle S^4 \rangle}{\langle S^2 \rangle^2} \\ &t \to t = \frac{\langle S^4 \rangle^2}{\langle S^2 \rangle^4}. \end{aligned} \tag{40}$$

Therefore the two relations (23) keep on being satisfied and, to the order we computed absence of chaos, it is stable against soft-spin distribution. It is important to remember that the constant terms in (23) belong to the expansion of τ_{12} in powers of τ_1, τ_2, which is the same as in the SK model because of the rescaling of the order parameter. Let us comment that, for technical reasons, to consider the free energy to the fifth order allows us to obtain a result valid up to sixth order. We showed that the relation $p(1) = p_d$ implies exactly $\epsilon = 0$; this statement can be checked at all orders by using the SP equations for $p(1)$ and p_d [24]. Substituting the first

into the second one obtains

$$\epsilon = 2(p_d - p(1))\big(a_d - a(1) + \tfrac{4}{3}(p^2(1) + p_d^2 + p(1)p_d) + 2v(p_d^{(2)} - p^{(2)}(1)) - 2yp_d^{(2)}\big). \quad (41)$$

So ϵ is equal to the difference between $p(1)$ and p_d multiplied by a second-order factor so that, if we prove that $p(1) = p_d$ to order n, the free-energy difference will be zero, at least at order $n + 4$.

7. Conclusion

The present paper's main results has been to phrase a consistent analytical picture for absence of chaos in temperature in mean-field spin glasses and to collect evidence through direct computation that this picture actually holds.

We made use of a model of two coupled systems that. through the replica trick, can be phrased as a variational problem with an order parameter analogous to the standard one. We proposed a set of solutions of the model and showed that they are intrinsically non-chaotic due to their structure. The actual existence of these solutions is therefore the main problem. It has been checked to fifth order in the reduced temperature for the SK model and for its generalization with soft-spin distribution.

A particular set of cancellations in the SP equations turned out to hold through subtle relations between the parameters of the models; however, it was proven that they hold exactly at all orders in the SK model in connection with the zero-loop expansion of replica field-theory [16].

Other cancellations were checked only at finite order: they constitute the main open problem. In this respect we recall that the failure of some cancellations, even at a very high order, will destroy the whole construction, leading to a dramatic change in the physical picture. The solutions will become unstable; therefore, even near the critical temperature, the function $P(q_{T1T2})$ will reduce in the thermodynamic limit to a δ function centred at zero.

We also considered the possibility that the non-trivial structure we propose for the solutions be incompatible with the SP equations. In the introduction we phrased the problem through the notion of structural consistency. It is a necessary condition to solve the SP equations. We have proven that the solutions we propose satisfy this condition at all orders in the expansion in powers of the order parameter. Furthermore, this condition imposes that the corrections one obtains, considering higher powers in the expansion in the reduced temperature, change the solutions quantitatively but not qualitatively. In particular, the relation $p(x) = p_d$ for $x > x_c$ holds exactly; the approximation is in the exact value of x_c. Also the starting points of the intermediate plateaux of $q_1(x)$ and $q_2(x)$ at all orders are exactly equal to x_c.

Though these solutions were obtained in the context of coupled systems, they are solutions of the SP equations for uncoupled systems too. Therefore they enter the average over solutions in equation (14) implying a non-trivial $P(q_{T1T2})$.

However, we warn the reader that nothing like the relation $P(q) = \mathrm{d}x/\mathrm{d}q$ can be written for these solutions. This relation follows from the possibility of substituting the average over solutions in (13) with a summation over the indices evaluated on the standard Parisi solution. Instead, in the case of two or more systems this is impossible because the various solutions cannot be obtained one from the other by a permutation of the indices [24].

What is implied by our results is that $P(q_{T1T2})$ has a non-zero support from zero to a maximum value $p_{\max} = q_{1\max} + \mathrm{O}(\tau^2)$, where $q_{1\max}$ is the self-overlap of the states at the higher temperature. The small positive corrections to this last relation remain small at any temperatures; actually, it can be proven that $p_{\max} = q_{1\max}$ holds at all temperatures at the level of accuracy of the Parisi–Toulouse approximation [27].

On general grounds it is reasonable that, like the standard Parisi solution, these solutions encode much more information than the mere value of the free energy; actually, it seems that they lead to a quantitative description of the bifurcation picture for the free-energy landscape discussed in the introduction [27].

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Appendix

In this appendix we report the solutions as evaluated by solving the SP equations to fourth order in the reduced temperatures for a given constraint q_c intended to be of the same order of magnitude of the self-overlap of the states (i.e. τ_1 or τ_2). From these expressions the maximum value of q_c can be readily obtained as the value for which the first and second plateaux of the function corresponding to the system at the higher temperature merge:

$$\text{for} \quad 0 \leqslant x \leqslant q_c \left(\frac{u}{\omega} + \frac{3uv - 8t\omega}{2\omega^3}(\tau_1 + \tau_2) \right)$$

$$q_1(x) = \left(\frac{\omega}{u} - \frac{(4\tau_1 + 2\tau_2)uv - (12\tau_1 + 4\tau_2)t\omega}{2u^2\omega} \right) x$$

$$p(x) = \left(\frac{\omega}{u} - \frac{3(\tau_1 + \tau_2)uv - 8(\tau_1 + \tau_2)t\omega}{2u^2\omega} \right) x$$

$$\text{for} \quad q_c \left(\frac{u}{\omega} + \frac{3uv - 8t\omega}{2\omega^3}(\tau_1 + \tau_2) \right) \leqslant x \leqslant q_c \left(\frac{2u}{\omega} + \frac{5uv - 12t\omega}{\omega^3}\tau_1 + \frac{uv - 4t\omega}{\omega^3}\tau_2 \right)$$

$$q_1(x) = \left(1 - \frac{uv - 4t\omega}{2u\omega^2}(\tau_1 - \tau_2) \right) q_c \qquad p(x) = q_c$$

$$\text{for} \quad q_c \left(\frac{2u}{\omega} + \frac{5uv - 12t\omega}{\omega^3}\tau_1 + \frac{uv - 4t\omega}{\omega^3}\tau_2 \right) \leqslant x \leqslant \frac{2u\tau_1}{\omega^2} + \frac{\tau_1^2}{\omega^4}(u(2u + 3v + 2y) + 6uv - 16t\omega)$$

$$q_1(x) = \left(\frac{\omega}{2u} - \frac{6uv - 16t\omega}{4u^2\omega}\tau_1 \right) x \qquad p(x) = q_c$$

$$\text{for} \quad \frac{2u\tau_1}{\omega^2} + \frac{\tau_1^2}{\omega^4}(u(2u + 3v + 2y) + 6uv - 16t\omega) \leqslant x \leqslant 1$$

$$q_1(x) = \frac{\tau_1}{\omega} + \frac{\tau_1^2(2u + 3v + 2y)}{2\omega^3} - \frac{\tau_1^3}{24\omega^5}(-48u^2 - 144uv - 108v^2 - 64uy - 144vy - 48y^2 + 120s\omega + 208t\omega + 48z\omega) \qquad p(x) = q_c.$$

The function $q_2(x)$ can be obtained from $q_1(x)$ exchanging τ_1 with τ_2. Given that the higher temperature is T_1, the maximum value of q_c for which the solutions exist works out to be

$$q_{c\,\max} = q_{1\,\max} + \frac{16t\omega - 6uv}{4u^2\omega^2}(\tau_2 - \tau_1)\tau_1.$$

The above expression is valid to second order in the reduced temperatures since the total number of matrix elements with value q_c is of order one due to the equality $p(1) = p_d$. Therefore,

the first order maximum overlap is equal to the self-overlap q_{EA} of the system at the higher temperature with positive higher-order corrections.

This quantity is accessible to direct measurement in numerical simulations and a comparison can be made, provided that the trivial order-parameter rescaling which has led us from (7) to (17) is taken into account. It is important to notice that this rescaling does not change the qualitative behaviour of the solutions, in particular the three functions $q_1(x)$, $q_2(x)$ and $p(x)$ remain different in the small-x region.

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